SESSION TWO
ANGLES AND THEIR MEASUREMENT

Outcomes

- Understand what angles are, and be able to identify acute, right, and obtuse angles
- Determine the size of angles of common regular polygons
- Understand the process of measuring an angle using wedges and simplified protractors
- Use benchmarks for angles: 30°, 45°, 60°, 90°
- Use commercial protractors to measure angles
- Determine the sum of the angles of a triangle in different ways
- Determine the sum of the angles of a quadrilateral

Time and Overview

5 minutes	The session starts by forming angles with straws and bobby pins. This will give participants a basic understanding of what angles are. Participants will form small angles and angles that are bigger than a right angle.
20–30 minutes	Participants will determine the size of the angles of equilateral triangles, hexagons, other pattern block shapes, and shapes formed by combining these shapes. They will use their knowledge that the angle of a square is 90°.
25–30 minutes	Use 30° wedges to illustrate how units of measurement are repeated to determine the size of an angle.
10 minutes	Use a simplified 15° protractor to measure angles.
10–15 minutes	Use construction triangles to develop benchmarks for angles 30°, 45°, 60°, 90°.
10 minutes	Measure angles using a commercial protractor.
5–10 minutes	Determine the sum of the angles of a triangle by ripping a paper triangle and rearranging the angles.
10 minutes	Measure the three angles of a triangle with a protractor.
15–20 minutes	Use tiles of congruent tiles to determine the sum of the angles of a triangle.
10–15 minutes	Find the sum of the angles of different quadrilaterals.

Materials

Facilitator	Transparencies (Eng. & Spanish)
• Overhead pattern blocks • Set of transparent 30° wedges • Transparent 15° protractor • Transparent construction triangles 30-60-90, and 45-45-90 • Transparent protractor • Set of transparent triangle tiles	*BLMs 14.1-2: Measuring Angles with 30º Wedges* *BLMs 15.1-3: Measuring Angles with a 15º Protractor* *BLMs 16.1-2: Benchmarks for Angles* *BLMs 17.1-5: Measuring Angles with a Protractor*

Materials

Participant	Handouts (English & Spanish)
• Two straws and two bobby pins per participant • Pattern blocks for each group • Set of 30˚ wedges per participant • One 15˚ protractor per participant • Set of 30-60-90 and 45-45-90 construction triangles per participant (not beveled, and not rounded corners) • One standard protractor per participant • One ruler per participant • One set of four congruent quadrilaterals per participant • Calculators • Colored pencils	**One per participant for class** *BLMs 12.1-2: Angles with Straws and Bobby Pins* *BLMs 13.1-4: Angles with Pattern Blocks* *BLMs 14.1-2: Measuring Angles with 30º Wedges* *BLMs 15.1-3: Measuring Angles with 15º Protractor* *BLMs 16.1-2: Benchmarks for Angles* *BLMs 17.1-5: Measuring Angles with a Protractor* **(copy 18.2-3 single sided and copy 18.6 on cardstock)** *BLMs 18.1-6: Sum of the Angles of a Triangle* *BLM 19: Additional Activity for the Sum of the Angles in a Triangle* *BLM 20: The Sum of the Angles in a Quadrilateral*

Activities

Preparation of Classroom	Notes
1. Prepare handouts for distribution for class activities.	Handouts can either be placed on the tables before the session begins or passed out at the beginning of each activity.
2. Place the name cards from last class near the front of the room where participants can easily find them.	
3. Have participant materials, manipulatives, and handouts on the tables.	
4. Prepare a set of four congruent quadrilaterals for each participant.	It is easy to cut four congruent shapes using a paper cutter by cutting four sheets of paper at the same time. Make sure the shapes do not slide when cutting them.

Angles with Straws and Bobby Pins (5 minutes)	
Materials and handouts: • Straws and bobby pins • BLMs 12.1-2: Angles with Straws and Bobby Pins **Opening Activity** 1. Before measuring angles, it is important that participants have experience forming angles of different sizes, that is, of variable openings. Straws joined by bobby pins are convenient for participants to manipulate so that they can increase gradually the size of the angle.	Other materials such as pipe cleaners can also be used to join the straws and form angles.

Activities

Angles with Straws and Bobby Pins (continued)	Notes
2. Each participant should have two straws and two bobby pins. Ask them to interlock the bobby pins and insert them into the straws as shown in the handout. The straws should be free to rotate around the vertex so formed. 3. An important aspect to emphasize is that the size of the angle depends on the amount of opening, not on how long the sides are. Different sizes of straws can be used to emphasize that the length of the sides is not important. 4. Participants can use the corner of a rectangular sheet of paper as their benchmark for right angles. Explain that angles that have a smaller opening than the right angle are called acute. Angles that have a greater opening than a right angle are called obtuse. Before using specific numbers to describe angles, participants should be able to tell that acute angles are small, and that obtuse angles are big.	
Angles with Pattern Blocks (20-30 minutes)	
Materials and handouts: • Pattern Blocks • Calculator (optional) • BLMs 13.1-4: Angles with Pattern Blocks **Activity 1** 1. In the approach taken in these activities, first participants manipulate angles that they know well, such as right angles, which are 90°. Ask participants to put together two squares next to each other to form a rectangle. With two squares together, they can realize that a straight line can be formed by two 90° angles. Therefore the measure of a stretched out angle is 180°. Ask participants to join three squares to form a 270°. Emphasize that the angle being measured is formed by the three angles of the squares joined together, not the missing 90° angle. Ask participants to join four squares around a common vertex. By joining four squares they will see that a full turn is 360°. It is convenient that participants trace the indicated angles with their fingers. 2. It is advisable to let participants also act out four turns of 90°. For example they could start facing north, then turn 90° (counterclockwise) so that they face west, then another 90° turn to face south, a third one to face east, and finally a fourth turn to face north again. With an 180° turn participants will face the opposite direction. When acting out the turns, participants need to realize that the sides of the angles are given by imaginary lines determined by the direction their eyes were pointing at each stage.	Having a calculator available may help some of the participants focus on the angles rather than on the computations.

Activities

Angles with Pattern Blocks (continued)	Notes
Activity 1 (continued) 4. Once participants know that there are 360° in a complete turn, they can use this fact to figure out what is the angle measure for other pieces in the pattern blocks, such as the equilateral triangle (with angles of 60°) and the regular hexagon (with angles of 120°). **Activity 2** Participants should remember that an equilateral triangle has three equal sides and three equal angles. Ask participants to fit six equilateral triangles around a point. Ask them to determine the size of the angles in an equilateral triangle, and explain their reasoning. A common explanation is that six angles together form a complete turn that is 360°. Therefore, each angle has to be 360° ÷ 6 = 60°. **Activity 3** Remind participants that a regular hexagon has six equal sides and six equal angles. Ask participants to place three regular hexagons around a common point. Ask them to determine the size of the angles of a regular hexagon. Most participants will divide 360 ÷ 3 to obtain 120° as the answer. Participants may also use two equilateral triangles to cover one of the angles of the hexagon. Two angles of 60° are needed to cover one angle of the hexagon. In some cases, a few participants will not see the connection immediately between the number of pieces needed to fill the whole turn and the size of the pieces. The instructor may guide them by asking appropriate questions, such as: • *How many pieces did you use? (six)* • *How many equal angles are together? (six)* • *How much is the whole turn? (360˚)* • *How much does each angle measure? (360˚ ÷ 6 = 60˚)* **Activity 4** Ask participant to determine the angles in the blue rhombus and the tan rhombus. Ask them to justify their thinking. Participants may use different strategies to determine the angles of the other pieces. One approach is to form the shapes around a common vertex, count how many pieces fit, and divide 360° by the corresponding number. That way they can see that the light tan rhombus has acute angles of 30° (12 tan rhombuses complete a turn)	

Activities

Angles with Pattern Blocks (continued)	Notes
and that the blue rhombus has 60˚ angles (6 blue rhombuses complete a turn). They can also overlap shapes to compare with angles they already know. For example they can overlap two tan rhombuses on top of the triangle. They will see that the measure of the angle has to be 60˚ ÷ 2 = 30˚. **Activity 5** Ask participants to determine the size of the angles formed by joining two pattern blocks together as shown in the handout (a square and a blue rhombus; a blue rhombus and a tan rhombus; five tan rhombuses; two tan rhombuses and a blue rhombus). In order to figure out the angles formed by adjacent shapes, participants will need to add angles together. This additive property is a fundamental property of measurement. **Activity 6** Ask participants to give a convincing argument that angles of the pieces mentioned in the handouts (two regular hexagons and two equilateral triangles; three equilateral triangles and two squares) do indeed add to 360˚. Participants should add the measures of the angles they already know to verify.	
Measure Angles with 30º Wedges (10 minutes)	
Materials and handouts: • Set of 30º wedges • BLMs 14.1-2: Measuring Angles with 30º Wedges 1. The basic process of measuring involves selecting a unit, repeating the unit so that the object to be measured is divided into equal subdivisions, and finally the units are counted to produce a number that represents the measure of the object. In the case of angles, however, the unit of measurement, 1˚, is too small to allow a hands-on repetition of units in the same way we can do with units of length or area. In these activities wedges of 30˚ are used so that participants can iterate these bigger units. 2. Provide participants with a set of 30˚ wedges. Illustrate how wedges can be placed together to measure bigger angles. Make sure the vertices of the angles are together. Ask participants to measure the angles given in the handout. The first examples are angles that are whole	The use of wedges provides a kinesthetic, sensory feeling of iterating the unit when filling an angle. Participants need time to experience this feeling at their leisure.

Activities

Measure Angles with 30º Wedges (continued)	Notes
multiples of 30° (60°, 120°, 180°, 30°, 90°, 150°). Then an example is given where one and a half of a wedge is needed (45°).	
Measure Angles with a 15º Protractor (10 minutes)	
Materials and handouts: • 15º protractor • BLMs 15.1-3: Measuring Angles with a 15º Protractor 1. Illustrate on the overhead projector how the protractor 15° is used. To measure angles it is important that the protractor is placed so that the central point is at the vertex of the angle that will be measured and that one of the end lines of the protractor coincides with one of the sides of the angle. Of course, to fit given angles the protractor will sometimes need to be rotated (because of convenience in making the drawings, the protractor is shown here with the base parallel to the lower border of the page). 2. Some participants may use different ways to measure the angles that may work in some particular cases (for example setting a 60° angle in the middle and counting that it covers 4 angles of 15°. While it is important to acknowledge alternative strategies, the instructor needs to make sure all participants learn also the standard way to place the protractor, with one end line coinciding with one side of the angle. This method is the one that will be used with the standard protractor later. **Activity** 1. Ask participants to use the 15° protractor to measure the angles given in the previous handout (60°, 120°, 180°, 30°, 90°, 150°, 45°). 2. Then ask them to measure the angles given. The angles given are multiples of 15° (75°, 105°, 135°).	Before using the standard protractor it is important that participants use a protractor that is less busy, and where the process of counting units set side by side is more explicit. In some instances participants will not be familiar with the protractor at all. They will need enough time with this simpler protractor.
Benchmarks for Angles (10-15 minutes)	
Materials and handouts: • Set of 30-60-90 and 45-45-90 construction triangles • BLMs 16.1-2: Benchmarks for Angles	

Activities

Benchmarks for Angles (continued)	Notes
1. Participants need to become familiar with angles that can be used as benchmarks. Having a mental image of triangles formed by those benchmark angles is a convenient way to have these benchmarks in their minds to compare with other angles. Very helpful are the construction triangles. By using construction triangles participants will be able to have angles such as 30˚, 45˚, 60˚, and 90˚ as references to estimate other angles. 2. Before using the angles in the construction triangles, participants need to know what is the measure of each angle. Some of the participants may not have previous experience with construction triangles. They need time to experiment with them and figure out the size of the angles on their own. For example, putting together the bigger angles of the two triangles participants will be able to form a straight line, so that they will see that both are right triangles. The triangles in two sets can be combined to determine the other angles. **Activity 1** Ask participants to describe benchmark angles (0˚, 30˚, 45˚, 60˚, 90˚, 120˚, 135˚, 150˚, and 180˚) in their own words. They may relate the angles in the triangles to known angles. For example, two isosceles construction triangles form a square; therefore the angles in this triangle are 45˚, 45˚, 90˚. Of the other kind, two triangles form an equilateral triangle, therefore its angles are 30˚, 60º, and 90˚. **Activity 2** Ask participants to use the construction triangle to measure the given angles (75˚, 60˚, 150˚, 135˚). The given angles can be obtained as a combination of at most two angles of the construction triangles. **Activity 3** Ask participants to estimate the size of the given angles using the construction triangles. Participants should give an interval for the value of the angle. They may also say to what value is the measure of the angle closer. For example, in the first case, the angle is between 30˚ and 45˚, closer to 45˚. The second is between 75˚ and 90˚, closer to 75˚. The third is between 60˚ and 75˚, closer to 75˚. The fourth is between 90˚ and 105˚, closer to 105˚.	

Activities

Measuring Angles with a Protractor (10 minutes)	Notes
Materials and handouts: • Protractors, rulers, and pencils • BLMs 17.1-5: Measuring Angles with a Protractor This lesson involves learning to use (or remembering how to use) a protractor with all its subdivisions. In addition of the difficulty of using the protractor due to the smallness of the unit, the many numbers present confuse some participants. Participants should realize that there are two numberings. One that goes from 0° to 180°, right to left and another in the reverse order. They should also realize when each scale is more convenient to use depending on the position of the angle to be measured. Participants need plenty of time to get confident using this tool to measure angles.	Some participants, especially if they did not have the opportunity to learn how to use the protractor when they were children, may feel that the time in class to practice measuring with such a complex tool is not enough. One participant borrowed a set of triangles and protractor to take home and practice some more.
Activity 1. Three important points on the protractor Participants need to get familiar with three important points on the protractor, the central mark, and the marks corresponding to 0° and 180°. Ask them to find the marks corresponding to 0° and 180° on both sides of the protractor. Draw their attention to the line segment on the protractor joining the marks on opposite sides of the protractor. Let participants find the mark for the midpoint of this segment (this we will call the central mark).	
Activity 2. Scale that goes left to right The first angles given in the handout are easier to measure by using the scale that has the 0° mark on the left. Let participants make the 0° mark on the left of the protractor coincide with the horizontal leg of the angle. They should notice the scale going from 0° on the left to 90° on the top, and then continuing to 180° on the right. One way to know whether they are using the correct scale is to remember that if they are measuring an angle smaller than a right angle, the measure should be less than 90°. If they are measuring an angle bigger than a right angle, then they should get a measure bigger than 90°.	To measure angles with small sides, it is convenient to extend the sides before measuring. Let participants use a ruler and a pencil to extend the sides.
Activity 3. Scale that goes right to left The second set of angles is easier to measure using the 0° mark on the right of the protractor. This time the scale goes from 0° on the right, to 90° on the top, to 180° on the left. **Activity 4. Using either scale** Let participants decide what scale to use to measure the angles in the last set.	

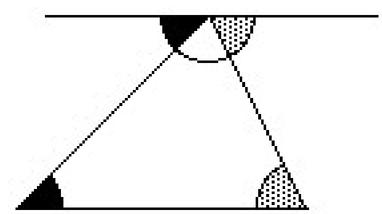
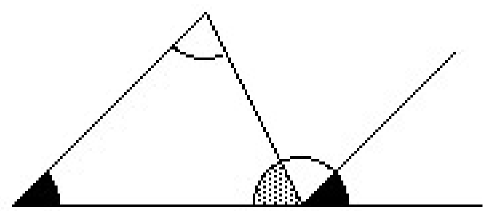
Activities

Sum of the Angles of a Triangle (5-10 minutes)	Notes
Materials and handouts: • Cardstock copy of the pattern on BLM 18.6 (to make a set of triangle tiles) • Scissors • BLMs 18.1-6: Sum of the Angles of a Triangle • BLM 19: Additional Activity for the Sum of the Angles in a Triangle **Activity 1. Tear and Join** 1. In the first activity everybody receives a different triangle. Participants should be aware that different kinds of triangles are used: some triangles with all acute angles, some with an obtuse angle, a few with a right angle; isosceles, scalene, and equilateral; different sizes. The triangles can be cut in advance with a paper cutter and it only takes a few minutes to cut a set for a class. 2. Ask participants to identify and label the angles of the triangle, so that they can recognize them after the triangle is torn apart. 3. Ask participants to tear the triangle with their hands in three parts so that each part has one of the angles. Model for the class. Ask them to join the three angles around a common point. In a few cases, participants will need to be directed as to how to join the angles. If they have problems understanding, they usually will see how someone else is doing it and then be able to join their own angles. 4. Ask them what they observe with the three angles together. Let them describe it using their own words. Some will describe how the arcs that mark the angles form half a circle; others will say the angles form a straight line. Participants should look around to see that everybody was able to form a straight line by joining the three angles of the triangle. The instructor can also illustrate on the overhead projector. 5. Remind participants that a straight line can be formed by two right angles, so that the angle measure of the straight line is 180°. Participants should state the result that the sum of the angles of a triangle is therefore 180°.	It is important that the paper is torn rather than cut so that no new angles are introduced.

Activities

Sum of the Angles of a Triangle (continued)	Notes
Activity 2. Measuring the Angles of a Triangle 1. Ask participants to measure the angles of the triangles in the handout and write the corresponding measures in the table. Make sure that all participants are using the protractor in a correct way and that they are reading the proper scale. When participants read the wrong scale, the instructor may ask whether the angle given is smaller or bigger than 90°, and the participant will realize he or she is using the wrong numbers. 2. It is important for participants to realize that there will be some measurement error involved and that the sum of the three angles may not always be exactly 180°. Participants need to feel comfortable with the fact that small measurement errors are present, so if the sum is 178° or 183° that is all right. Some participants will need explicit validation from the instructor that a sum that is off by 2° is actually correct, given the measurement instruments we are using. However, if the accumulated error is too big, then participants will need to measure again. **Activity 3. Triangular Tiles** 1. A set of triangular tiles for each participant needs to be prepared. Have participants construct their own sets by using the pattern on BLM 18.6 and cutting out the triangles. 2. Ask participants to observe that all the triangles are congruent to each other, that corresponding sides have the same length, and corresponding angles are equal. They should observe also that angles that are equal have been marked with the same color. 3. Ask participants to use the triangular tiles to form a straight path, with edges parallel to each other. 4. When forming the straight path, make sure that the participants have the tiles facing up, that is, so that the colored angles are visible. Make sure everybody has formed as straight path. Usually there at two ways to form the path, one wider that the other. Both are appropriate. The instructor can show both kinds on the overhead projector.	Some participants will be tempted to adjust their measurements so that the sum is exactly 180°. The instructor may want to discuss that whenever real measures are involved there is some measurement error.

Activities

Sum of the Angles of a Triangle (continued)	Notes
Activity 3 (continued) 5. Ask participants to look at the angles that are together at one of the sides of the straight path. Notice that they have one of each color. Ask them: "What is the sum of these three angles?" 6. Now let them look at one of the triangles. They will notice that it has angles that are equal to the corresponding angles that form the straight line. What can they say about the sum of the angles in the triangle compared to the sum of the three angles at the side of the path that form the straight line? The instructor can contrast the kind of reasoning in this activity with the first one (torn triangle), where only empirical verification was used. In this activity participants will have to do some deductive thinking also to reach the conclusion (therefore this is a good activity to facilitate the transition to level 3 of geometric thinking). They do not use angles from the same triangle to form the straight line; instead they use two angles that are congruent to the desired angles. 7. This activity does also provide a bridge to some of the traditional ways to prove that the sum of the angles in a triangle is 180˚. For those participants who are ready to make the connection, the instructor may point to the corresponding part in the straight path . 8. Ask participants to use the triangular tiles to form a double straight path. Ask them to look at one of the vertices in the center of the path. Ask them to describe the six angles around a point. Ask them what the sum of the six angles is. 9. If they do not bring it up, ask them to describe the pattern of colors of the angles. What can they say about angles that opposite to each other?	 Proof using two pairs of alternate interior angles  Proof based using corresponding angles and alternate interior angles
Extension: The Sum of the Angles in a Quadrilateral (10-15 minutes)	
Materials and handouts: • Set of four congruent quadrilaterals • BLM 20: The Sum of the Angles in a Quadrilateral **Activity 1** 1. A set of four congruent arbitrary quadrilaterals is needed for each participant.	It is easy to cut four congruent shapes using a paper cutter by cutting four sheets of paper at the same time. Make sure the shapes do not slide when cutting them.

Activities

<table>
<tr><th>Extension: The Sum of the Angles in a Quadrilateral (continued)</th><th>Notes</th></tr>
<tr><td>
Activity 1 (continued)
2. Ask participants to color the angles of one of the quadrilaterals using four different colors. Then ask them to color the angles in the other quadrilaterals with the same color as the corresponding congruent angle.

3. Ask participants to place four copies of the quadrilateral around a common vertex, so that all colors are present. Participants will need to rotate two of the quadrilaterals. What can they say about the sum of the angles in an arbitrary quadrilateral?

Activity 2
1. Another method to find the sum of the angles in a quadrilateral is by realizing that the quadrilateral can be decomposed into two triangles.

2. Ask participants to use the fact that the sum of the angles in each of the triangles is 180˚ to obtain the sum of the angles in the quadrilateral. The sum is 2 x 180˚ = 360˚.

3. By cutting a quadrilateral into two triangles a very powerful process is illustrated. That is, solve a problem in terms of a simpler problem that they have solved before.
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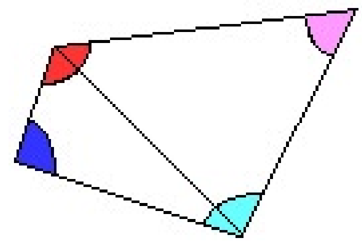
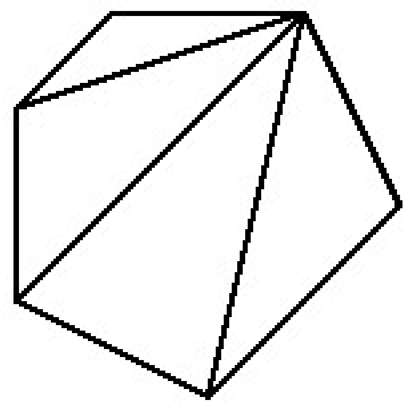

The angles of the two triangles together form the angles of the quadrilateral.


It is also a process that can be applied to a polygon with any number of sides.
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<tr><th>Take Home Activities</th><th></th></tr>
<tr><td>Have participants take home any activities or extensions not completed in class.</td><td></td></tr>
<tr><th>Preparation for the Next Session</th><th></th></tr>
<tr><td>Collect name cards for use in the next sessions.</td><td></td></tr>
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