SESSION FOUR
VOLUME AND GEOMETRIC REPRESENTATION OF OPERATIONS

Outcomes

- Understand the volume formula for a rectangular solid
- Compute surface area of rectangular solids
- Explore the relation of the surface areas of solids of similar shapes
- Explore the relation of volume of solids of similar shapes
- Relate the volume of a pyramid to the volume of a rectangular solid with the same base and same height
- Relate the volume of a cone to the volume of a cylinder with the same base and same height
- Use area models to represent properties of numbers and illustrate arithmetic procedures
- Use paper folding to illustrate several geometrical concepts, such as lines, perpendicular lines, angles bisector, etc.

Overview

This session has three distinct parts: volume, geometric representations of arithmetic operations, and paper folding. Some parts may be incorporated into other sessions.

Time

15-20 minutes	In the first part participants use wooden cubes to understand the formula for the volume of a rectangular solid.
10–15 minutes	Participants compute surface area of rectangular solids and look for patterns.
10-15 minutes	Participants explore how the surface area changes as the solid increases but preserving its shape. They explore how the volumes of similar solids are related.
10-15 minutes	Through a demonstration, participants relate the volume of a pyramid and a cone to the volume of a rectangular solid and a cylinder.
20–30 minutes	In the second part, area models are used to illustrate properties of numbers and operations, such as commutative and distributive properties, and to represent visually computational algorithms.
20–30 minutes	In the third part, paper folding is used as an alternative to represent lines and to find properties about lines, angles, and other geometrical objects.

Materials

Facilitator	Transparencies (Eng. & Spanish)
• Demonstration Transparent Solid set (volume of pyramid, rectangular solid, cone, and cylinder) • Base ten blocks set for overhead projector • Square tiles for overhead projector • Big squares of paper (tamale paper)	*BLMs 27.1-2: Shapes*
Participant	**Handouts (English & Spanish)**
• Transparent Solid set (volume of pyramid, rectangular solid, cone, and cylinder) - if available • A set of base ten blocks per table • A set of wooden cubes per table • Square tiles • Patty paper, regular paper, and grid paper • Adding machine tape • A cut-out right triangle	**One per participant for class** *BLMs 28.1-2: Volume* *BLM 29: Volume of a Pyramid and Cone* *BLMs 32.1-3: Geometric Representations of Arithmetical Operations* *BLMs 33.1-4: Paper Folding Geometry* **One per participant for class and home** *BLM 30: Cube* (copy on cardstock) *BLM 31: Pyramid* (copy on cardstock)

Activities

Preparation of Classroom	Notes
1. Prepare handouts for distribution for class activities. 2. Place the name cards from last class near the front of the room where participants can easily find them. 3. Have participant materials and handouts on the tables. 4. Prepare before class begins, a paper cut-out of a right triangle for each participant.	Handouts can either be placed on the tables before the session begins or passed out at the beginning of each activity.
Volume (1 hour and 5 minutes)	
Materials, transparencies, and handouts: • Wooden cubes and transparent solids • BLMs 27.1-2: Shapes • BLMs 28.1-2: Volume **Opening Activity (15-20 minutes)** 1. Ask participants to build a two by three by four rectangular solid using the unit cubes. Ask them how many cubes did they use. Ask them how can they count the number of cubes in a systematic way. 2. Suggest slicing the cube into layers. Ask them to represent the number of cubes in each layer. 3. Different teams will slice the rectangular solid in different ways. Some will have two layers of 12 cubes. Others will have 3 layers of 8 cubes. Some others will have 4 layers of 6 cubes. **Volume Formulas** The volume of a three dimensional solid is given by the number of unit cubes it contains. The volume can be found by counting the unit cubes one by one. For some solids, there are more efficient ways of counting the cubes. These ways lead to volume formulas. However, it is important for participants to realize that those formulas are not the definition of volume. The volume of a solid can be found even if we do not have a formula at hand. **Activity 1 (10-15 minutes)** 1. A formula for the volume of a rectangular solid is area of base times height. Ask participants to justify the volume of a rectangular box, Volume = area of the base times height. 2. In a rectangular solid that has the height of one unit, the number of cubes is exactly the same as the same as the number of unit squares contained in its base. The number of cubes and the area are given by the same number. There is	Participants should have the opportunity to build the rectangular solid using wooden cubes. The term *capacity* is also used for volume, especially when dealing with fluids. From a mathematical point of view both terms are equivalent. In everyday practice, however, some people prefer one term to the other depending on the context.

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Volume (continued)	Notes
a one to one correspondence between the number of cubes in a layer and the number of unit squares in the base. However, it is important to keep in mind the difference between the two kinds of objects. When dealing with volume, we are counting unit cubes. 3. When a rectangular solid has a height of more than one unit, we can think of it as being composed by layers, each layer one unit thick. Each layer has the same number of cubes, which is the same as the number of unit squares of the base. The number of layers is given by the height of the solid. So when we multiply the area of the base times the height, we are indeed counting cubic units in an efficient way. We multiply the number of cubes in a layer by the number of layers. **Activity 2. An alternative formula for the volume.** 1. Often there are participants who remember the formula *length × width × height* for the volume of a rectangular solid. However, in most cases they have little understanding of how the formula is obtained, or how come we obtain a volume when multiplying the lengths of three linear segments. 2. Ask participants to justify the formula *length × width × height* for the volume of a rectangular solid. 3. When using the formula $V = a \times b \times c$ to find the volume of a rectangular solid, it is also important to realize that this is just another way to count unit cubes efficiently. We can think in terms of rows and layers of cubes. The number a represents the number of cubes in a row. The number b represents the number of rows in a layer, and c represents the number of layers. 4. Alternatively, we can think of the number $a \times b$ as the number of cubes in the first layer. This number is also the number of unit squares on the base. So we can see that the two formulas length × width × height and area of base × height are equivalent. **Activity 3. Volume and associativity of multiplication.** 1. When we want to find the product of three numbers a, b, and c, it does not matter if we multiply the first two numbers first and then the result by the third, or if we multiply the first number by the product of the other two.	

Activities

Volume (continued)	Notes
Activity 3 (continued) That is, we have $a \times (b \times c) = (a \times b) \times c$. This is the associative property for multiplication. Of course, the technical term is not what is important, but the concept behind. 2. Ask participants to describe how a rectangular solid made of wooden cubes can be sliced in different ways to illustrate the associative property of multiplication. 3. For the given example, when the product of three numbers represents the volume of a rectangular brick, participants should slice the brick in two different ways to represent both sides of the equation and identify what each of the terms of this equation represents. The upper part of the figure represents the left side of the equation $3 \times (2 \times 4) = (3 \times 2) \times 4$, the lower part represents the right side. In the left side of the equation, 2×4 is the number of cubes in a slice, *3* is the number of slices. In the right side, 3×2 is the number of cubes in slice, and *4* are the number of slices. **Activity 4. Increasing towers example**. 1. Ask participants to build towers with wooden cubes as shown in the handouts. Ask them to find the surface area of each tower of cubes (not including the bottom). Let them organize the information in a table, describe the relationship between surface area and the number of cubes in their own words, notice patterns, and describe the relation with an equation. 2. There are different ways to describe the growth of the surface area in terms of the number of cubes used in a tower. Also there are different ways to describe the relationships between numbers in the table. Some participants will focus on the increment on the right side, a constant increment of *4*. Others will relate the number on the right with the corresponding number on the left. It is important that the number patterns are related to parts of the tower. **Activity 5. Surface and volume of similar figures. (10-15 minutes)** 1. When all dimensions of a figure change in the same proportion, volume and surface area change in very different ways. Compute the surface area and the volume of a $2 \times 3 \times 5$ rectangular box. If each of the dimensions of	

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Volume (continued)	Notes
the box is multiplied by *2*, we obtain a new box, *4 × 6 × 10*. • *What is the surface area of the new box?* • *What is its volume?* Justify your answer in at least two different ways. Notice that the surface area increased by a factor of 4, whereas the volume increased by a factor of 8. **Activity 6. Volume of a pyramid. (10-15 minutes)** 1. Of course, not all solids are rectangular. Nevertheless, we can find a formula for their volume by comparing them to solids of known volume. It is important to provide participants with the opportunity to do this experimentally. There are commercially available sets of transparent solids that can be filled with seeds or water. They include a cube and a pyramid that has the same base area and the same height as the cube. The experiment can be done easily with beans or other small seeds. If you have a sink available, water works fine too. If only one set of the transparent solids is available, this activity works well also as a teacher demonstration. 2. Let participants predict how many times the pyramid can be filled and its content poured into the cube. A common answer is two. Some participants, as well as prospective teachers, predict this because that is the relation of a triangle to the rectangle with the same base and the same height. The cross section of the pyramid is indeed a triangle, so it is easy to be mislead. Some participants are surprised that indeed the content of the pyramid can be poured three times into the cube. 3. After the experiment or demonstration, the formula for the volume of the pyramid can be introduced, $V = B \times h / 3$. Participants will have at least an empirical reason why we divide by 3. **Activity 7. Volume of a cone.** 1. The volume of the cone can be directly connected to the volume of the cylinder with the same base area and the same height by doing an experiment similar to the previous activity. Ask participants to predict how many times the volume of the cone will fit into the volume of the cylinder. Most participants will predict three times, based on their previous experience with the pyramid.	If the commercial set is not available, handouts (BLM 30 & 31) of a cube and pyramid are provided so that participants can build their own. Thin but sturdy and rigid cardboard works best or the handout could be copied on cardstock. These handouts also need to be reproduced to take home, so participants can do the experiment at home. For some participants this becomes evident after pouring the first pyramid; the cube will be 1/3 filled. Other participants however emphatically ask for the experiment to be completed.

Activities

Volume (continued)	Notes
Activity 7 (continued) 2. It is interesting to notice that cones are often used as containers for food, such as ice cream cones, and glasses for shakes. To the eye, the cone appears to have a bigger volume than it really has. 3. The volume of the cylinder is given by multiplying the area of the base times its height. Participants need to know that the area of a circle can be found, so a prerequisite to do this part is finding the area of a circle.	
Geometric Representations of Arithmetic Operations (20-30 minutes)	
Materials and handouts: • Square tiles and base ten blocks • BLMs 32.1-3: Geometric Representations It is very important that people develop meanings for arithmetic operations. Everyday problems in context provide an important source of meaning for arithmetic operations. Geometrical representations provide additional meaning for the operations, and provide connections between arithmetic and other fields. Geometric representations can provide an important support in the transition between arithmetic and algebra. **Area Models** 1. When we use rectangles to represent numbers, each unit square represents 1. We can count the total number of unit squares in a rectangle in a more efficient way by counting the number of squares in a row and then multiplying by the number of rows. 2. In addition, the sides of the rectangle also represent numbers. In the case of a rectangle, the number that represents the length is also the number of unit squares in a row. The width of the rectangle represents the number of rows. **Activity 1. Commutative Property** 1. Participants should describe the efficient way to compute the total number of unit squares of a rectangle first by an expression in plain English, such as "three rows with five squares each gives 15 squares", and then by a symbolic equation such as $3 \times 5 = 15$. Some participants may express the same situation as $5 \times 3 = 15$. What is important is that they know what the numbers represent. 3 is the number of rows, 5 is the number of squares per row. Other participants may describe the rectangle as "five columns	

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Geometric Representations of Arithmetic Operations (continued)	Notes
with three unit squares each". The second rectangle can be described as "five rows with three squares each", or as "three columns with five squares each". 2. The crucial part of this activity is that participants see that one rectangle can be transformed into the other just by rotating it. They can rotate them in their heads, or they may prefer to rotate the page. Younger children may benefit from working with cardboard rectangles that they can actually rotate and overlap. 3. Finally, participants should describe the situation with an equation $3 \times 5 = 5 \times 3$. Some participants may need help to grapple with this way of using the "=" sign. Many are used to interpret "=" as a command to perform the operation to the left, rather than a symbol indicating equality between two numeric expressions. 4. Ask participants to construct their own examples of commutative property using square tiles. Let them describe the situation using their own words, and using an equation. **Activity 2. Distributive Property** 1. It is convenient to illustrate the distributive property with small numbers first. Participants need to realize that this is a property of multiplication and addition of numbers, and it is not dependent on our base ten system. The distributive property can be illustrated by the addition of two rectangles with the same height to form a longer rectangle with the same height. 2. Ask participants to describe how the distributive property is illustrated by the rectangles given in the handouts. A thick line divides each rectangle. Let them describe the area of each rectangle by a product and by the sum of products corresponding to the parts. 3. Ask participants to use square tiles or a grid to illustrate the identities given in the handouts. $(5 + 3) \times 3 = 5 \times 3 + 3 \times 3$ $(10 + 2) \times 3 = 10 \times 3 + 2 \times 3$ **Activity 3. Distributive Property in the Multiplication of Two-digit Numbers** 1. In the usual algorithms used in multiplication, distributive property is used in conjunction with place value, we distribute preferably according to the different place	 Base ten blocks are very helpful to represent the quantities explicitly.

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Geometric Representations of Arithmetic Operations (continued)	Notes
values. For example, to multiply *4 × 16* we can multiply *4 × 10* and *4 × 6*, and add the two results. The product 4 × 16 can also be represented by a rectangle. The other two products can also be represented by corresponding rectangles. 2. We can describe the equality as *4 × (10 + 6) = 4 × 10 + 4 × 6*. Ask participants to use base ten blocks to build a rectangle to represent the product *4 × 16*. Ask them to identify the parts of the rectangle that correspond to the partial answers *4 × 10* and *4 × 6*. When multiplying a two-digit number by another two-digit number, the distributive property is used repeatedly. The geometrical representation can help participants see all the implicit steps. For a common algorithm taught in schools, participants should identify where each of the intermediates values are represented in the corresponding rectangle. This will help them slow down and reflect on a process that many times has been automated almost completely to the point that people performing the computation do not always know what the different steps in the process mean. For example, in the algorithm: 16 × 14 64 160 224 64 is represented as the two smaller rectangles in the upper part, 160 is represented by the two bigger rectangles below. Some participants find it illuminating to expand the algorithm even more to make more explicit how the different products are obtained. 3. Ask participants to use base ten blocks to form a rectangle that represents the product *12 × 13*. Ask them to identify the parts of the rectangle that correspond to the intermediate steps in the algorithm they use.	Several participants expressed their joy for understanding for the first time why the algorithm they had learned works. For example, with this representation, it becomes clear that the second line is a multiple of ten, because we are multiplying by a number in the tens place. Often, participants learned to “move” the second line in the partial result one place to the left when dealing with the next digit, without realizing why.
Paper Folding Geometry (20-30 minutes)	
Materials and Handouts: • Patty paper, regular paper, and adding maching tape • A paper cut-out right triangle for each participant • BLMs 33.1-4: Paper Folding Geometry Patty paper works very well for participants. The creases are easy to see. Also usually it is easier to draw segments and angles on one side of the patty paper.	Patty paper sometimes it is too small for the instructor to demonstrate or model. A solution is to use bigger sheets of paper to illustrate the folds, or to wander around the room to get close to all the participants to make sure they see how the fold is made.

Activities

Paper Folding Geometry (continued)	Notes
Activity 1. A straight line 1. Ask participants to mark two points P and Q on the patty paper. Ask them to fold over a portion of a sheet of paper so that point P is placed on top of point Q. They should hold the two points P and Q tightly together using a finger and the table, and crease the paper with the other hand. Then they extend the crease in both directions to form a straight line. 2. Let them mark a new arbitrary point R on the crease. What can they say about the distance of R to P and to Q? Let them fold along the crease again to verify. Ask them to use other points along the crease to convince themselves that the points on the crease are equidistant from P and Q. 3. Ask them to draw the segment connecting the points P and Q. Ask them: • *What is the angle formed by the drawn segment and the crease?* • *Where does the crease intersect the segment?* 4. Tell participants that a line that is perpendicular to a segment and divides the segment in two equal parts is called the perpendicular bisector of the segment. Let them draw a segment and make a crease that divides the segment in half and is perpendicular to it. 5. The instructor can help participants become aware that the different creases are obtained due to mathematical properties. In the beginning, let participants get familiar with the folding process first. Later, they can come back and think about the mathematics behind it. For example, in the first fold, by holding two points of the paper tight together with two fingers and then creasing, is the same as having two fixed points and then finding all the points that are equidistant from those two points. The result is the perpendicular bisector of the segment joining the two points. **Activity 2. A straight line perpendicular to a given straight line** 1. To form a line perpendicular to a given line, the process is very similar. By folding in such a way that the original line overlaps on itself we ensure that we have two points on the same line on top of each other. The perpendicular bisector of this segment (which is part of the line) will be formed.	Participants need to feel that it is OK to make mistakes. Have additional unused sheets ready in case someone folds the wrong crease. Rather than trying to correct, it is often easier to start again with a clean sheet. Make sure that all participants are able to follow every step of the instructions. It can be very difficult to manage a class where people are at different stages. Some participants may have difficulty seeing the segment, because the points are not marked on the paper, only indicated by the position of the fingers. Participants can repeat the process (with a new sheet) by first marking two points on the paper, and folding so that the points overlap. Participants may want to draw the segment between the two points. When the paper is folded, they will see that each point on the crease is equidistant from the two endpoints of the segment.

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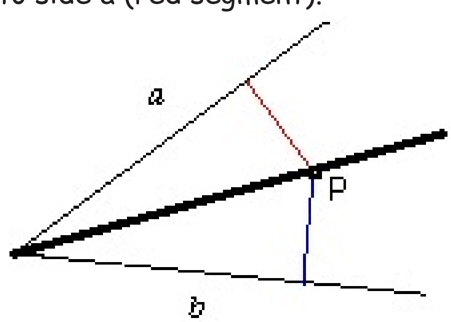
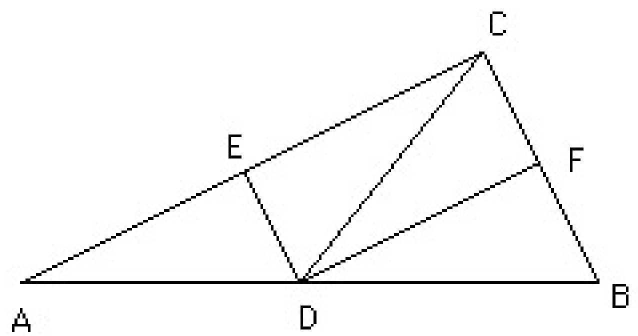
Paper Folding Geometry (continued)	Notes
2. Ask participants to draw or fold a line AB on patty paper. Let them fold the sheet over so that a segment of the given line AB is folded over onto itself. Ask them to hold the lines together with the fingers of one hand and the table, and form a crease with the other hand. Let them observe the angle between the original line and the new crease. **Activity 3. A line parallel to a given straight line** 1. Ask participants to construct EF perpendicular to AB, and then construct CD perpendicular to EF. What can they say about AB and CD? 2. To obtain a line parallel to the given line, participants can use the property that two lines perpendicular to the same line are parallel. **Activity 4. Bisect an angle** 1. Ask participants to trace an angle ACB (that is, the vertex of the angle is at C). Let them fold and crease the paper so that the legs CA and CB of the given angle coincide on top of each other. Let them make sure the crease goes through the vertex of the angle. Then ask them to open the paper and look at the two smaller angles formed. What can they say about them? 2. The instructor can also help participants see that the mathematical property behind bisecting an angle by folding is that any point on the angle bisector is equidistant from the two sides. In figure 1, point P is on the angle bisector. The distance to side b (length of blue segment) is equal to the distance to side a (red segment).  **Activity 5. The midpoint of the hypotenuse** 1. In advance, cut a right triangle for each participant. Ask them to find the midpoint of the hypotenuse. Remind them this is the longest side of the right triangle, the one opposite to the right angle.	

Activities

Paper Folding Geometry (continued)	Notes
2. Ask them to show, by folding the triangle, that the midpoint of the hypotenuse of a right triangle is at the same distance from all three vertices of the triangle. 3. Because the distance of the midpoint of the hypotenuse to each of the vertices is the same, it is the center of the circumscribed circle of a right triangle. Drawing the segment DC on the triangle will help participants see what segments have the same length.  	
Activity 6. A 60˚ Angle 1. Give participants a rectangular piece of paper. Tell them to label each vertex A, B, C, and D on the paper. Ask them to fold the line that is parallel to two borders and is equidistant from them. They can do this by overlapping AC on top of BD. Let them open the piece of paper. (Label G on the paper.) 2. Ask them to fold vertex A onto the median so that the resulting crease GB passes through B. Let them denote by J the position of A on EF. (Label J on the paper.) Ask them to provide convincing evidence or prove that angle ABJ is 60º 3. Once participants have located point J on the mid parallel as indicated in the activity, they should be encouraged to draw segments JA and JB to help their reasoning. 4. The reason a 60˚ angle is obtained by the fold described is that any point of the mid parallel is equidistant from the two vertices A and B, and that by making A be on this mid parallel we ensure that we find a point whose distance to these vertices is the same as the length of segment AB. Therefore we obtain an equilateral triangle; its angles are 60˚. This approach is convenient for participants who are level 3 of their development of geometrical thinking, it may be too difficult for some participants.	Left without any guidance, even people who are quite strong in mathematics find providing a proof for this activity challenging, mainly because the paper folding setting is not something they are familiar with.

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Paper Folding Geometry (continued)	Notes
5. Some participants may prefer to verify empirically that the angle is indeed 60˚. One way is to fold BD onto BG (when A is on the mid parallel) and see that the angles GBA and JBD are equal. That is, a 90˚ angle has been divided into three equal parts. The instructor should acknowledge also this empirical way to give convincing arguments, and not try to force participants to use the more advanced level of thinking if they are not ready. However, it is very important that all participants reach at least this empirical stage and go beyond simple perception. **Activity 7. A regular pentagon from a strip of paper** 1. Each participant will use a long strip of paper of constant width (adding machine paper works very well). Ask participants to tie a simple knot. Direct them to tighten the knot gently and crease it flat as they do it. 2. Participants should convince themselves that all angles are equal and that all sides are congruent. They can see this by measuring or overlapping, and also by using the symmetry of the figure	The knot should be formed by using the full width of the strip.
Take Home Activities	
Have participants take home any actvities from the handouts that were not completed.	
Preparation for the Next Session	
Collect name cards for use in the next sessions.	